17 Poincaré–Andronov–Hopf bifurcation

I already mentioned that there are no regular methods to study the limit cycles of the systems on the plane. Probably, one of the most important approaches, together with the Poincaré–Bendixson theory, is the Poincaré–Andronov–Hopf bifurcation[1], which is the only genuinely two dimensional bifurcation (i.e., it cannot be observed in systems of dimension 1), which can occur in generic two dimensional autonomous systems depending on one parameter (*co-dimension* one bifurcation).

Consider the system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \alpha), \quad \boldsymbol{x}(t) \in U \subseteq \mathbf{R}^2 \tag{1}$$

that depends on a scalar parameter $\alpha \in \mathbf{R}$.

Definition 1. *A bifurcation of an equilibrium of system* (1)*, for which a pair of purely imaginary eigenvalues* $\lambda_{1,2} = \pm \mathrm{i}\omega_0$, $\omega_0 > 0$, *appears, is called the Poincaré–Andronov–Hopf bifurcation, or the bifurcation of the birth of a limit cycle.*

Example 2. Consider the system

$$\begin{aligned} \dot{x} &= \alpha x - y - x(x^2 + y^2), \\ \dot{y} &= x + \alpha y - y(x^2 + y^2). \end{aligned} \tag{2}$$

This system has the equilibrium $\hat{\boldsymbol{x}}_0 = (0, 0)$ for all parameter values α, and the Jacobi matrix of (2) evaluated at $\hat{\boldsymbol{x}}_0$ is

$$\boldsymbol{A} := \boldsymbol{f}'(\hat{\boldsymbol{x}}_0) = \begin{bmatrix} \alpha & -1 \\ 1 & \alpha \end{bmatrix}.$$

The eigenvalues of $\boldsymbol{A}$ are

$$\lambda_{1,2}(\alpha) = \alpha \pm \mathrm{i}.$$

Introducing complex variable $z = x + \mathrm{i}y$, (2) can be rewritten in the complex form

$$\dot{z} = (\alpha + \mathrm{i})z - z|z|^2.$$

Using the exponential form of the complex numbers $z = re^{\mathrm{i}\theta}$, I can rewrite (2) in the polar coordinates as

$$\dot{r} = r(\alpha - r^2), \quad \dot{\theta} = 1.$$

The last system can be easily analyzed since the equations are decoupled. The first equation always has the equilibrium $\hat{r}_0 = 0$, and, if $\alpha > 0$, another equilibrium $\hat{r}_1 = \sqrt{\alpha}$. The linear analysis shows that $\hat{r}_0 = 0$ is asymptotically stable if $\alpha < 0$ and unstable if $\alpha > 0$. Note that if $\alpha = 0$ we cannot analyze the stability of $\hat{r}_0 = 0$ by the linear approximation, however, the trivial equilibrium of $\dot{r} = -r^3$ is asymptotically stable. For $\alpha > 0$ another equilibrium $\hat{r}_1$ appears. The second equation describes the counterclockwise rotation with constant speed. Superposition of these two behaviors yields the bifurcation diagram of system (2) (see the figure), which shows that for $\alpha > 0$ an asymptotically stable unique limit cycle appears.

Math 484/684: Mathematical modeling of biological processes by Artem Novozhilov
e-mail: artem.novozhilov@ndsu.edu. Fall 2015.

[1]V.I. Arnold writes in his textbook “Geometrical methods in the theory of ordinary differential equations”: “Considered above theorem was known essentially to Poincaré. An explicit formulation and proof were given by A.A. Andronov. [...] R.Tom, who I thought this theory in 1965, was advocating it under the name Hopf bifurcation.”

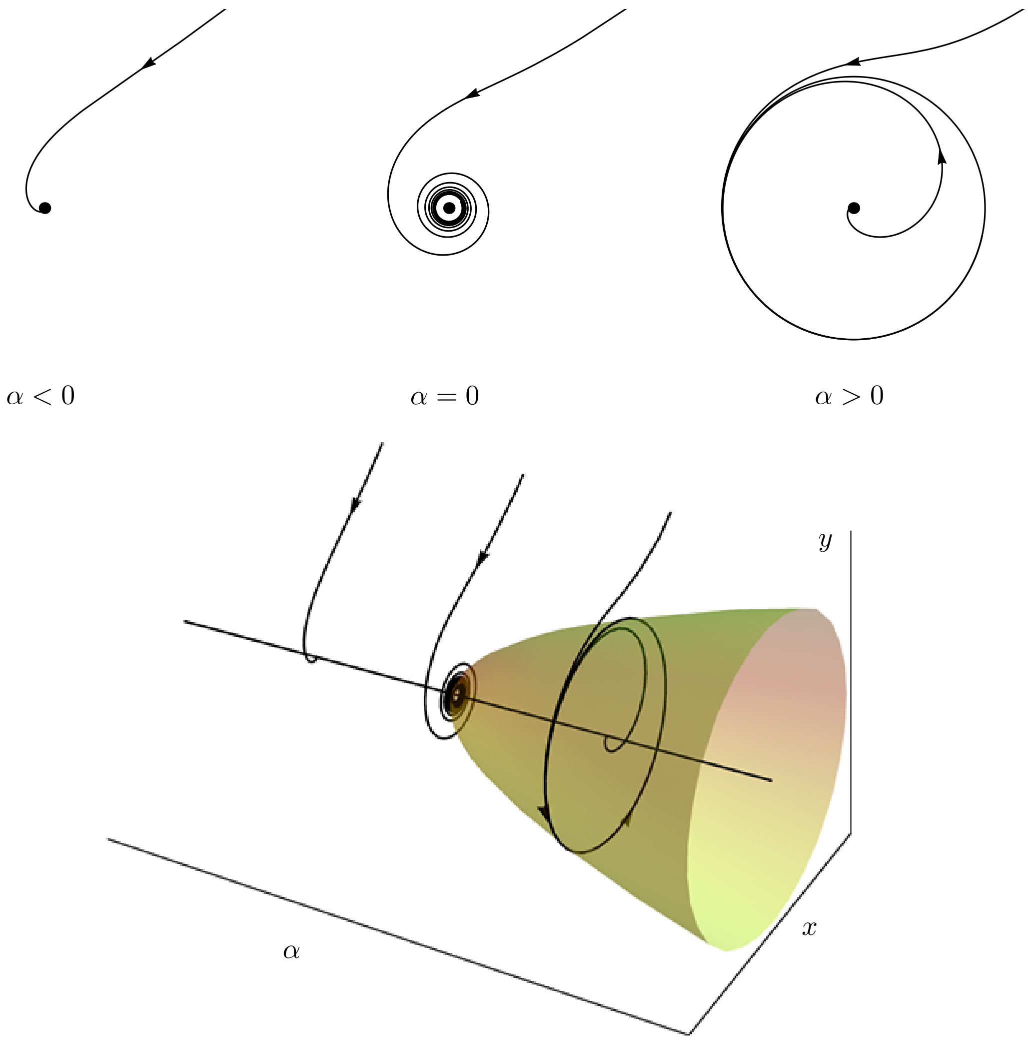

Figure 1: Supercritical Poincaré–Andronov–Hopf bifurcation

System

$$
\begin{aligned}
\dot{x} &= \alpha x - y + x(x^2 + y^2), \\
\dot{y} &= x + \alpha y + y(x^2 + y^2),
\end{aligned}
\tag{3}
$$

can be analyzed in a similar way. This system also has Poincaré–Andronov–Hopf bifurcation for $\alpha = 0$, the difference is that the limit cycle exists for $\alpha < 0$ and it is unstable. For $\alpha > 0$ there is no limit cycle, and $\hat{r}_0 = 0$ is unstable. Note that for $\alpha = 0$ non-hyperbolic equilibrium $\hat{r}_0$ is unstable (see the figure).

Bifurcation in (2) is called *supercritical*, the limit cycle exists for positive parameter values ("after" the bifurcation), whereas the bifurcation in (3) is called *subcritical*. In the former case the unique stable equilibrium is replaced with a unique asymptotically stable limit cycle of a small amplitude $\sqrt{\alpha}$, and the system stays in a neighborhood of $\hat{r}_0 = 0$. This is so-called *soft* or *non-catastrophic* loss of stability. In the latter case, the basin of attraction of $\hat{r}_0$ is bounded by the unstable limit cycle for negative α,

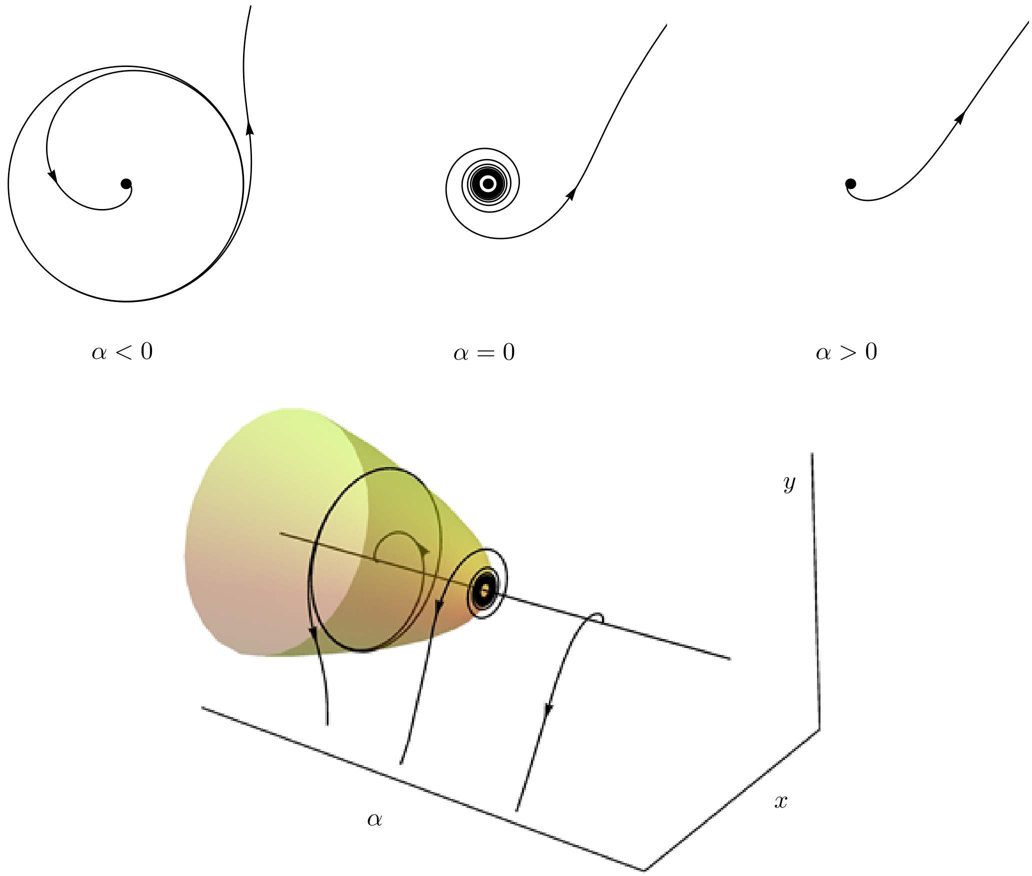

Figure 2: Subcritical Poincaré–Andronov–Hopf bifurcation

and if α becomes positive, the system leaves any neighborhood of the origin. This is so-called *hard* or *catastrophic* loss of stability. The type of the Poincaré–Andronov–Hopf bifurcation (soft or hard) is determined by the stability of the trivial equilibrium at the bifurcation parameter value.

It turns out that the situation in the example above appears in many different systems of the form (1). Here is a general statement without proof.

Theorem 3. *Any system* (1) *that has an equilibrium* $\hat{\boldsymbol{x}}$ *for the parameter values* $|\alpha - \alpha_b| < \epsilon$ *for some* $\epsilon > 0$*, whose linearization has eigenvalues* $\lambda_{1,2}(\alpha) = \mu(\alpha) \pm \mathrm{i}\omega(\alpha)$ *such that* $\mu(\alpha_b) = 0$, $\omega(\alpha_b) = w_0 > 0$*, and satisfying the following conditions*

$$\frac{\mathrm{d}\mu}{\mathrm{d}\alpha}(\alpha)\bigg|_{\alpha=\alpha_b} \neq 0, \tag{4}$$

and

$$L_1(\alpha_b) \neq 0, \tag{5}$$

experiences the Poincaré–Andronov–Hopf bifurcation. The bifurcation is supercritical if $L_1(\alpha_b) < 0$ *and subcritical if* $L_1(\alpha_b) > 0$.

Remark 4. The statement of the theorem includes the condition $L_1(\alpha_b) \neq 0$ for the first Lyapunov values L_1 at the bifurcation parameter value α_b. Moreover, the sign of $L_1(\alpha_b)$ actually determines the type of the bifurcation. However, I never explained how to actually calculate $L_1(\alpha_b)$. This is a nontrivial computational problem and I skip it in these lectures. The interested reader should consult an extremely readable account in Kuznetsov's textbook[2]. The type of the Poincaré–Andronov–Hopf bifurcation can be inferred if the stability of the equilibrium $\hat{\boldsymbol{x}}$ at the bifurcation value $\alpha = \alpha_b$ can be analyzed (e.g., with the help of the Lyapunov functions). Anyway, appearance of purely imaginary eigenvalues that cross the imaginary axis with non-zero speed should indicate that it is possible to have a limit cycle somewhere close.

Example 5. To illustrate this theorem, consider the following predator–prey system

$$\begin{aligned}\dot{N} &= rN\left(\frac{N}{L_p}-1\right)\left(1-\frac{N}{K_p}\right)-aNP,\\ \dot{P} &= -cP+dNP,\end{aligned}$$

where all the parameters are assumed to be positive, and $L_p < K_p$. This system can be put in dimensionless form as

$$\begin{aligned}\dot{x} &= x(x-l)(K-x)-xy,\\ \dot{y} &= -\gamma y+xy,\end{aligned}$$

where $0 < l < K$ and $\gamma > 0$. It is possible to have up to four equilibria:

$$\hat{\boldsymbol{x}}_0 = (0,0), \quad \hat{\boldsymbol{x}}_1 = (l,0), \quad \hat{\boldsymbol{x}}_2 = (K,0), \quad \hat{\boldsymbol{x}}_3 = \big(\gamma, (\gamma-l)(K-\gamma)\big),$$

and $\hat{\boldsymbol{x}}_3 \in \mathbf{R}^2_+$ if and only if $l < \gamma < K$. The Jacobi matrix for our system is

$$\boldsymbol{f}'(\boldsymbol{x}) = \begin{bmatrix}(2x-l)(K-x)-x(x-l)-y & -x\\ y & x-\gamma\end{bmatrix}.$$

Analysis of the eigenvalues yields that $\hat{\boldsymbol{x}}_0$ is an asymptotically stable node, $\hat{\boldsymbol{x}}_1$ is an unstable node if $l > \gamma$, otherwise it is a saddle with the unstable manifolds on x-axis, $\hat{\boldsymbol{x}}_2$ is an asymptotically stable node if $K < \gamma$, otherwise it is a saddle with stable manifolds on x-axis. The Jacobi matrix at $\hat{\boldsymbol{x}}_3$ takes the form

$$\boldsymbol{f}'(\hat{\boldsymbol{x}}_3) = \begin{bmatrix}\gamma(K+l-2\gamma) & -\gamma\\ \gamma(K+l-\gamma)-Kl & 0\end{bmatrix},$$

which implies that

$$\operatorname{tr}\boldsymbol{f}'(\hat{\boldsymbol{x}}_3) = \gamma(K-2\gamma+l), \quad \det\boldsymbol{f}'(\hat{\boldsymbol{x}}_3) = \gamma\big(\gamma(K-\gamma+l)-lK\big).$$

Therefore, if $\gamma > (l+K)/2$ then $\hat{\boldsymbol{x}}_3$ is asymptotically stable. If $\gamma = (l+K)/2 =: \gamma_b$, then

$$\operatorname{tr}\boldsymbol{f}'(\hat{\boldsymbol{x}}_3) = 0, \quad \det\boldsymbol{f}'(\hat{\boldsymbol{x}}_3) = \frac{(K+l)(K-l)^2}{8},$$

[2]Kuznetsov, I.A. (1998). Elements of applied bifurcation theory (Vol. 112). Springer.

and I have a non-hyperbolic equilibrium with purely imaginary eigenvalues. It is easy to check the first condition for the Poincaré–Andronov–Hopf bifurcation:

$$\frac{\mathrm{d}\mu}{\mathrm{d}\gamma}(\gamma)\Big|_{\gamma-\gamma_b} = \frac{1}{2}\,\frac{\mathrm{d}\,\mathrm{tr}\,\boldsymbol{f'}(\hat{\boldsymbol{x}}_3)}{\mathrm{d}\gamma}\Big|_{\gamma=\gamma_b} = -K-l \neq 0.$$

Somewhat tedious calculations lead to

$$L_1(\gamma_b) = -\frac{1}{\omega_0}\frac{\sqrt{2}\sqrt{K+l}}{4(K-l)}\,,$$

which proves that the bifurcation is supercritical (note that $\omega_0 = \det \boldsymbol{f'}$), with appearance of a unique stable limit cycle (see the figure).

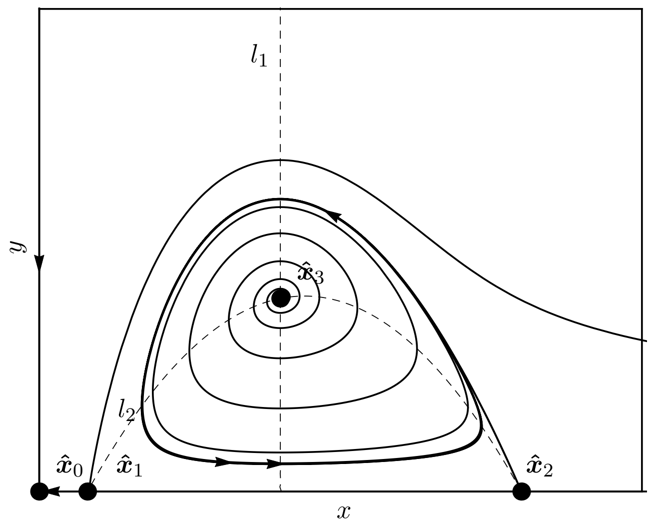

Figure 3: The limit cycle in Example 5